Lasry-Lions regularization and a Lemma of Ilmanen

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Let H be a Hilbert space. We define the following inf (sup) convolution operators acting on bounded functions $u : H \longrightarrow \mathbb{R}$:

$$T_t u(x) := \inf_y \big(u(y) + \frac{1}{t}\|y-x\|^2\big)$$

and

$$\breve{T}_t u(x) := \sup_y \big(u(y) - \frac{1}{t}\|y-x\|^2\big).$$

We have the relation

$$T_t(-u) = -\breve{T}_t(u).$$

Recall that these operators form semi-groups, in the sense that

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad \breve{T}_t \circ \breve{T}_s = \breve{T}_{t+s}$$

for all $t \geqslant 0$ and $s \geqslant 0$, as can be checked by direct calculation. Note also that

$$\inf u \leqslant T_t u(x) \leqslant u(x) \leqslant \breve{T}_t u(x) \leqslant \sup u$$

for each $t \geqslant 0$ and each $x \in H$. A function $u : H \longrightarrow \mathbb{R}$ is called k-semi-concave, $k > 0$, if the function $x \longrightarrow u(x) - \|x\|^2/k$ is concave. The function u is called k-semi-convex if $-u$ is k-semi-concave. A bounded function u is t-semi-concave if and only if it belongs to the image of the operator T_t, this follows from Lemma 1 and Lemma 3 below. A function is called semi-concave if it is k-semi-concave for some $k > 0$. A function u is said $C^{1,1}$ if it is Frechet differentiable and if the gradient of u is Lipschitz. Note that a continuous function is $C^{1,1}$ if and only if it is semi-concave and semi-convex, see Lemma 6. Let us recall two important results in that language:

Theorem 1. *(Lasry-Lions, [6]) Let u be a bounded function. For $0 < s < t$, the function $\breve{T}_s \circ T_t u$ is $C^{1,1}$ and, if u is uniformly continuous, then it converges uniformly to u when $t \longrightarrow 0$.*

Theorem 2. *(Ilmanen, [5]) Let $u \geqslant v$ be two bounded functions on H such that u and $-v$ are semi-concave. Then there exists a $C^{1,1}$ function w such that $u \geqslant w \geqslant v$.*

Our goal in the present paper is to "generalize" simultaneously both of these results as follows:

Theorem 3. *The operator $R_t := \breve{T}_t \circ T_{2t} \circ \breve{T}_t$ has the following properties:*

- *Regularization : $R_t f$ is $C^{1,1}$ for all bounded f and all $t > 0$.*
- *Approximation : If f is uniformly continuous, then $R_t f$ converges uniformly to f as $t \longrightarrow 0$.*

- *Pinching: If $u \geqslant v$ are two locally bounded functions such that u and $-v$ are k-semi-concave, then the inequality $u \geqslant R_t f \geqslant v$ holds for each $t \leqslant k$ if $u \geqslant f \geqslant v$.*

Theorem 3 does not, properly speaking, generalize Theorem 1. However, it offers a new (although similar) answer to the same problem: approximating uniformly continuous functions on Hilbert spaces by $C^{1,1}$ functions with a simple explicit formula.

Because of its symmetric form, the regularizing operator R_t enjoys some nicer properties than the Lasry-Lions operators. For example, if f is $C^{1,1}$, then it follows from the pinching property that $R_t f = f$ for t small enough.

Theorem 2, can be proved using Theorem 3 by taking $w = R_t u$, for t small enough. Note, in view of Lemma 3 bellow, that $R_t u = \check{T}_t \circ T_t u$ when t is small enough.

Theorem 3 can be somehow extended to the case of finite dimensional open sets or manifolds via partition of unity, at the price of loosing the simplicity of explicit expressions. Let M be a paracompact manifold of dimension n, equipped once and for all with an atlas $(\phi_i, i \in \Im)$ composed of charts $\phi_i : B^n \longrightarrow M$, where B^n is the open unit ball of radius one centered at the origin in $\mathbb{R}^n$. We assume in addition that the image $\phi_i(B^n)$ is a relatively compact open set. Let us fix, once and for all, a partition of the unity g_i subordinated to the open covering $(\phi_i(B^n), i \in \Im)$. It means that the function g_i is non-negative, with support inside $\phi_i(B^n)$, such that $\sum_i g_i = 1$, where the sum is locally finite. Let us define the following formal operator

$$G_t(u) := \sum_i \left[R_{ta_i}\big((g_i u) \circ \phi_i\big)\right] \circ \phi_i^{-1},$$

where $a_i, i \in \Im$ are positive real numbers. We say that a function $u : M \longrightarrow \mathbb{R}$ is locally semi-concave if, for each $i \in \Im$, there exists a constant b_i such that the function $u \circ \phi_i - \|.\|^2/b_i$ is concave on B^n.

Theorem 4. *Let $u \geqslant v$ be two continuous functions on M such that u and $-v$ are locally semi-concave. Then, the real numbers a_i can be chosen such that, for each $t \in]0,1]$ and each function f satisfying $u \geqslant f \geqslant v$, we have:*

- *The sum in the definition of $G_t(f)$ is locally finite, so that the function $G_t(f)$ is well-defined.*
- *The function $G_t f$ is locally $C^{1,1}$.*
- *If f is continuous, then $G_t(f)$ converges locally uniformly to f as $t \longrightarrow 0$.*
- *$u \geqslant G_t f \geqslant v$.*

Notes and Acknowledgements

Theorem 2 appears in Ilmanen's paper [5] as Lemma 4G. Several proofs are sketch there but none is detailed. The proof we detail here follows lines similar to one of the sketches of Ilmanen. This statement also has a more geometric counterpart, Lemma 4E in [5]. A detailed proof of this geometric version is given in [2], Appendix. My attention was attracted to these statements and their relations with recent progresses on sub-solutions of the Hamilton-Jacobi equation (see [4, 1, 7]) by Pierre Cardialaguet, Albert Fathi and Maxime Zavidovique. These authors also recently wrote a detailed proof of Theorem 2, see [3]. This paper also proves how the geometric version follows from Theorem 2. There are many similarities between the tools used in the present paper and those used in [1]. Moreover, Maxime Zavidovique observed in [7] that the existence of $C^{1,1}$ subsolutions of the Hamilton-Jacobi equation in the discrete case can be deduced from Theorem 2. However, is seems that the main result of [1] (the existence of $C^{1,1}$ subsolutions in the continuous case) can't be deduced easily from Theorem 2. Neither can Theorem 2 be deduced from it.

1 The operators T_t and $\check{T}_t$ on Hilbert spaces

The proofs of the theorems follow from standard properties of the operators T_t and $\check{T}_t$ that we now recall in details.

Lemma 1. *For each bounded function u, the function $T_t u$ is t-semi-concave and the function $\check{T}_t u$ is t-semi-convex. Moreover, if u is k-semi-concave, then for each $t < k$ the function $\check{T}_t u$ is $(k-t)$-semi-concave. Similarly, if u is k-semi-convex , then for each $t < k$ the function $T_t u$ is $(k-t)$-semi-convex.*

Proof. We shall prove the statements concerning T_t. We have

$$T_t u(x) - \|x\|^2/t = \inf_y \big(u(y) + \|y-x\|^2/t - \|x\|^2/t\big) = \inf_y \big(u(y) + \|y\|^2/t - 2x\cdot y/t\big),$$

this function is convex as an infimum of linear functions. On the other hand, we have

$$T_t u(x) + \|x\|^2/l = \inf_y \big(u(y) + \|y-x\|^2/t + \|x\|^2/l\big).$$

Setting $f(x,y) := u(y) + \|y-x\|^2/t + \|x\|^2/l$, the function $\inf_y f(x,y)$ is a convex function of x if f is a convex function of (x,y). This is true if u is k-semi-convex, $t < k$, and $l = k - t$ because we have the expression

$$f(x,y) = u(y) + \|y-x\|^2/t + \|x\|^2/l = (u(y) + \|y\|^2/k) + \Big\|\sqrt{\frac{l}{kt}}y - \sqrt{\frac{k}{lt}}x\Big\|^2.$$

□

Given a uniformly continuous function $u : H \longrightarrow \mathbb{R}$, we define its modulus of continuity $\rho(r) : [0,\infty) \longrightarrow [0,\infty)$ by the expression $\rho(r) = \sup_{x,e} u(x+re) - u(x)$, where the supremum is taken on all $x \in H$ and all e in the unit ball of H. The function ρ is non-decreasing, it satisfies $\rho(r+r') \leqslant \rho(r) + \rho(r')$, and it converges to zero in zero (this last fact is equivalent to the uniform continuity of u). We say that a function $\rho : [0,\infty) \longrightarrow [0,\infty)$ is a modulus of continuity if it satisfies these properties. Given a modulus of continuity $\rho(r)$, we say that a function u is ρ-continuous if $|u(y) - u(x)| \leqslant \rho(\|y-x\|)$ for all x and y in H.

Lemma 2. *If u is uniformly continuous, then the functions $T_t u$ and $\check{T}_t u$ converge uniformly to u when $t \longrightarrow 0$. Moreover, given a modulus of continuity ρ, there exists a non-decreasing function $\epsilon(t) : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\lim_{t\longrightarrow 0} \epsilon(t) = 0$ and such that, for each ρ-continuous bounded function u, we have:*

- *$T_t u$ and $\check{T}_t u$ are ρ-continuous for each $t \geqslant 0$.*
- *$u - \epsilon(t) \leqslant T_t u(x) \leqslant u$ and $u \leqslant \check{T}_t u \leqslant u + \epsilon(t)$ for each $t \geqslant 0$.*

Proof. Let us fix $y \in H$, and set $v(x) = u(x+y)$. We have $u(x) - \rho(|y\|) \leqslant v(x) \leqslant u(x) + \rho(\|y\|)$. Applying the operator T_t gives $T_t u(x) - \rho(y) \leqslant T_t v(x) \leqslant T_t u(x) + \rho(y)$. On the other hand, we have

$$T_t v(x) = \inf_z \big(u(z+y) + \|z-x)\|^2/t\big) = \inf_z \big(u(z) + \|z-(x+y)\|^2/t\big) = T_t u(x+y),$$

so that

$$T_t u(x) - \rho(\|y\|) \leqslant T_t u(x+y) \leqslant T_t u(x) + \rho(\|y\|).$$

We have proved that $T_t u$ is ρ continuous if u is, the proof for $\check{T}_t u$ is the same.

In order to study the convergence, let us set $\epsilon(t) = \sup_{r>0}(\rho(r) - r^2/t)$. We have

$$\epsilon(t) = \sup_{r>0} \big(\rho(r\sqrt{t}) - r^2\big) \leqslant \sup_{r>0} \big((r+1)\rho(\sqrt{t}) - r^2\big) = \rho(\sqrt{t}) + \rho^2(\sqrt{t})/4.$$

We conclude that $\lim_{t\longrightarrow 0}\epsilon(t) = 0$. We now come back to the operator T_t, and observe that

$$u(y) - \|y-x\|^2/t \geqslant u(x) - \rho(\|y-x\|) + \|y-x\|^2/t \geqslant u(x) - \epsilon(t)$$

for each x and y, so that

$$u - \epsilon(t) \leqslant T_t u \leqslant u.$$

$\square$

Lemma 3. *For each locally bounded function u, we have $\check{T}_t \circ T_t(u) \leqslant u$ and the equality $\check{T}_t \circ T_t(u) = u$ holds if and only if u is t-semi-convex. Similarly, given a locally bounded function v, we have $T_t \circ \check{T}_t(v) \geqslant v$, with equality if and only if v is t-semi-convex.*

Proof. Let us write explicitly

$$\check{T}_t \circ T_t u(x) = \sup_y \inf_z \big(u(z) + \|z-y\|^2/t - \|y-x\|^2/t\big).$$

Taking $z = x$, we obtain the estimate $\check{T}_t \circ T_t u(x) \leqslant \sup_y u(z) = u(z)$. Let us now write

$$\check{T}_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z \big(u(z) + \|z\|^2/t + (2y/t)\cdot(x-z)\big)$$

which by an obvious change of variable leads to

$$\check{T}_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z \big(u(z) + \|z\|^2/t + y\cdot(x-z)\big).$$

We recognize here that the function $\check{T}_t \circ T_t u(x) + \|x\|^2/t$ is the Legendre bidual of the function $u(x) + \|x\|^2/t$. It is well-know that a locally bounded function is equal to its Legendre bidual if and only if it is convex. $\square$

Lemma 4. *If u is locally bounded and semi-concave, then $\check{T}_t \circ T_t u$ is $C^{1,1}$ for each $t > 0$.*

Proof. Let us assume that u is k-semi-concave. Then $u = T_k \circ \check{T}_k u$, by Lemma 3. We conclude that $\check{T}_t \circ T_t u = \check{T}_t \circ T_{t+k} f$, with $f = \check{T}_k u$. By Lemma 1, the function $T_{t+k} f$ is $(t+k)$-semi-concave. Then, the function $\check{T}_t T_{t+k} f$ is k-semi-concave. Since it is also t-semi-convex, it is $C^{1,1}$. $\square$

2 Proof of the main results

Proof of Theorem 3: For each function f and each $t > 0$, the function $\check{T}_t \circ T_{2t} \circ \check{T}_t f$ is $C^{1,1}$. This is a consequence of Lemma 4 since

$$\check{T}_t \circ T_{2t} \circ \check{T}_t f = \check{T}_t \circ T_t(T_t \circ \check{T}_t f)$$

and since the function $T_t \circ \check{T}_t f$ is semi-concave.

Assume now that both u and $-v$ are k-semi-concave. We claim that

$$u \geqslant f \geqslant v \Longrightarrow u \geqslant T_t \circ \check{T}_t f \geqslant v \text{ and } u \geqslant \check{T}_t \circ T_t f \geqslant v$$

for $t \leqslant 1/k$. This claim implies that $u \geqslant \check{T}_t \circ T_{2t} \circ \check{T}_t f \geqslant v$ when $u \geqslant f \geqslant v$. Let us now prove the claim concerning $\check{T}_t \circ T_t$, the other part being similar. Since v is k-semi-convex, we have $\check{T}_t \circ T_t v = v$ for $t \leqslant k$, by Lemma 3. Then,

$$u \geqslant f \geqslant \check{T}_t \circ T_t f \geqslant \check{T}_t \circ T_t v = v$$

where the second inequality follows from Lemma 3, and the third from the obvious fact that the operators T_t and $\check{T}_t$ are order-preserving.

The approximation property follows directly from Lemma 2. □

PROOF OF THEOREM 4: Let a_i be chosen such that the functions $(g_i u) \circ \phi_i$ and $-(g_i v) \circ \phi_i$ are a_i-semi-concave on $\mathbb{R}^n$. The existence of real numbers a_i with this property follows from Lemma 5 below. Given $u \geqslant f \geqslant v$, we can apply Theorem 3 for each i to the functions

$$(g_i u) \circ \phi_i \geqslant (g_i f) \circ \phi_i \geqslant (g_i v) \circ \phi_i$$

extended by zero outside of B^n. We conclude that, for $t \in]0, 1]$, the function $R_{ta_i}((g_i f) \circ \phi_i)$ is $C^{1,1}$ and satisfies

$$(g_i u) \circ \phi_i \geqslant R_{ta_i}((g_i f) \circ \phi_i) \geqslant (g_i v) \circ \phi_i.$$

As a consequence, the function

$$\left[R_{ta_i}\big((g_i f) \circ \phi_i\big)\right] \circ \phi_i^{-1}$$

is null outside of the support of g_i, and therefore the sum in the definition of $G_t f$ is locally finite. The function $G_t(f)$ is thus locally a finite sum of $C^{1,1}$ functions hence it is locally $C^{1,1}$. Moreover, we have

$$u = \sum_i g_i u \geqslant G_t(f) \geqslant \sum_i g_i v = v.$$

□

We have used:

Lemma 5. *Let $u : B^n \longrightarrow \mathbb{R}$ be a bounded function such that $u - \|.\|^2/a$ is concave, for some $a > 0$. For each compactly supported non-negative C^2 function $g : B^n \longrightarrow \mathbb{R}$, the product gu (extended by zero outside of B^n) is semi-concave on $\mathbb{R}^n$.*

PROOF. Since u is bounded, we can assume that $u \geqslant 0$ on B^n. Let $K \subset B^n$ be a compact subset of the open ball B^n which contains the support of g in its interior. Since the function $u - \|.\|^2/a$ is concave on B_1 it admits super-differentials at each point. As a consequence, for each $x \in B^n$, there exists a linear form l_x such that

$$0 \leqslant u(y) \leqslant u(x) + l_x \cdot (y - x) + \|y - x\|^2/a$$

for each $y \in B^1$. Moreover, the linear form l_x is bounded independently of $x \in K$. We also have

$$0 \leqslant g(y) \leqslant g(x) + dg_x \cdot (y - x) + C\|y - x\|^2$$

for some $C > 0$, for all x, y in $\mathbb{R}^n$. Taking the product, we get, for $x \in K$ and $y \in B^n$,

$$u(y)g(y) \leqslant u(x)g(x) + (g(x)l_x + u(x)dg_x) \cdot (y - x) + C\|y - x\|^2 + C\|y - x\|^3 + C\|y - x\|^4$$

where $C > 0$ is a constant independent of $x \in K$ and $y \in B^n$, which may change from line to line. As a consequence, setting $L_x = g(x)l_x + u(x)dg_x$, we obtain the inequality

$$(gu)(y) \leqslant (gu)(x) + L_x \cdot (y - x) + C\|y - x\|^2 \tag{L}$$

for each $x \in K$ and $y \in B^n$. If we set $L_x = 0$ for $x \in \mathbb{R}^n - K$, the relation (L) holds for each $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. For $x \in K$ and $y \in B^n$, we have already proved it. Since the linear forms L_x, $x \in K$ are uniformly bounded, we can assume that $L_x \cdot (y-x) + C\|y-x\|^2 \geqslant 0$ for all $x \in K$ and $y \in \mathbb{R}^n - B^n$ by taking C large enough. Then, (L) holds for all $x \in K$ and $y \in \mathbb{R}^n$. For $x \in \mathbb{R}^n - K$ and y outside of the support g, the relation (L) holds in an obvious way, because $gu(x) = gu(y) = 0$, and $L_x = 0$. For $x \in \mathbb{R}^n - K$ and y in the support of g, the relation holds provided that $C \geqslant \max(gu)/d^2$, where d is the distance between the complement of K and the support of g. This is a positive number since K is a compact set containing the support of g in its interior. We conclude that the function (gu) is semi-concave on $\mathbb{R}^n$. □

For completeness, we also prove, following Fathi:

Lemma 6. *Let u be a continuous function which is both k-semi-concave and k-semi-convex. Then the function u is $C^{1,1}$, and $6/k$ is a Lipschitz constant for the gradient of u.*

Proof. Let u be a continuous function which is both k-semi-concave and k-semi-convex. Then, for each $x \in H$, there exists a unique $l_x \in H$ such that

$$|u(x+y) - u(x) - l_x \cdot y| \leqslant \|y\|^2/k.$$

We conclude that l_x is the gradient of u at x, and we have to prove that the map $x \longmapsto l_x$ is Lipschitz. We have, for eah x, y and z in H:

$$l_x \cdot (y+z) - \|y+z\|^2/k \leqslant u(x+y+z) - u(x) \leqslant l_x \cdot (y+z) + \|y+z\|^2/k$$

$$l_{(x+y)} \cdot (-y) - \|y\|^2/k \leqslant u(x) - u(x+y) \leqslant l_{(x+y)} \cdot (-y) + \|y\|^2/k$$

$$l_{(x+y)} \cdot (-z) - \|z\|^2/k \leqslant u(x+y) - u(x+y+z) \leqslant l_{(x+y)} \cdot (-z) + \|z\|^2/k.$$

Taking the sum, we obtain

$$\big|(l_{x+y} - l_x) \cdot (y+z)\big| \leqslant \|y+z\|^2/k + \|y\|^2/k + \|z\|^2/k.$$

By a change of variables, we get

$$\big|(l_{x+y} - l_x) \cdot (z)\big| \leqslant \|z\|^2/k + \|y\|^2/k + \|z-y\|^2/k.$$

Taking $\|z\| = \|y\|$, we obtain

$$\big|(l_{x+y} - l_x) \cdot (z)\big| \leqslant 6\|z\|\|y\|/k$$

for each z such that $\|z\| = \|y\|$, we conclude that

$$\|l_{x+y} - l_x\| \leqslant 6\|y\|/k.$$

□

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